$\boxed{\text{Extremos}}$

$f: \mathbb{R}^n \to \mathbb{R}$, de classe C^1.

$\nabla f(a) = 0 \to$ a é ponto crítico de f.
(candidato a extremo)

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Derivadas de ordem superior

Exemplo:

$f(x,y) = x^2 y^3$

$\boxed{\text{1ª ordem}}$

$\frac{\partial f}{\partial x}(x,y) = 2xy^3$

$\frac{\partial f}{\partial y}(x,y) = 3x^2y^2$

$\boxed{\text{2ª ordem}}$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = 2y^3$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) = 6xy^2$!!

Teorema de Schwarz $\to$ =

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = 6xy^2$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = 6x^2y$

<u>Notação</u>: $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x^2}$$

$$\longrightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial y \partial x}$$

$$\parallel$$

$$\longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial y^2}$$

//

Se f, $\frac{\partial f}{\partial x_j}$, $\frac{\partial^2 f}{\partial x_j \partial x_k}$, $j=1,2,\dots,n$, $k=1,2,\dots,n$ forem contínuas, diz-se que f é de <u>classe C^2</u>.

Teorema ou Lema de Schwarz:

Se $f:\mathbb{R}^n \to \mathbb{R}$ for de classe C^2

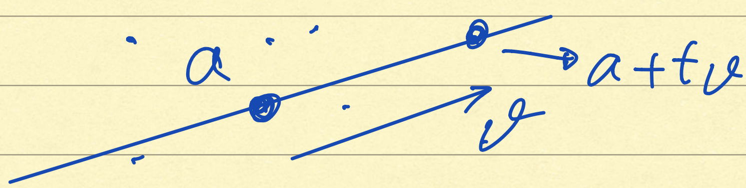
então $$\frac{\partial^2 f}{\partial x_j \partial x_k}(a) = \frac{\partial^2 f}{\partial x_k \partial x_j}(a)$$

$$j \neq k$$

$\mathbb{R}^2$: $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Classificar pontos críticos

$$h(t) = f(g(t)) \quad ; \quad g(t) = a + tv, \quad t \in \mathbb{R}$$



$$h(t) = f(g(t)) \quad ; \quad g(0) = a$$

$$h(0) = f(g(0)) = f(a)$$

$$\mathbb{R} \longleftarrow \mathbb{R}^n$$

Taylor: $h(t) = h(0) + h'(0)t + \frac{h''(0)}{2} t^2 + \underbrace{o(t^2)}_{\approx 0}$

$$\lim_{t \to 0} \frac{o(t^2)}{t^2} = 0.$$

$$h(t) - h(0) = \underbrace{\cancel{h'(0)t}}_{=0} + \frac{1}{2} \underbrace{h''(0)}_{\text{sinal}} t^2 + o(t^2)$$

$$\frac{h(t) - h(0)}{t^2} = \underbrace{\frac{1}{2} \boxed{h''(0)}}_{\text{sinal}} + \underbrace{\frac{o(t^2)}{t^2}}_{\approx 0}$$

O sinal de $h''(0)$ permite classificar o ponto crítico ($h'(0) = 0$).

$$h(t) = \underbrace{f(g(t))}\quad ; \quad g(t) = a + tv$$

$$t \in \mathbb{R}$$

$$g(0) = a$$

$$h'(t) = \nabla f(g(t)) \cdot g'(t)$$

composta

$$= \nabla f(g(t)) \cdot v = \underbrace{\sum_{k=1}^{n} \boxed{\frac{\partial f}{\partial x_k}(g(t))}\, v_k}$$

$$h'(0) = \nabla f(a) \cdot v$$

$$h'(0) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(a)\, v_k$$

$$h''(t) = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(g(t))\, v_j \right) v_k$$

$$h''(0) = \sum_{k=1}^{n} \sum_{j=1}^{n} \underbrace{\frac{\partial^2 f}{\partial x_j \partial x_k}}(a)\, v_j\, v_k$$

//

Sinal de $h''(0)$. ?

<u>Álgebra linear:</u> $A_{n\times n} = [A_{jk}]_{n\times n}$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} A_{jk}\, v_j \cdot v_k = v^T A v$$

↑ <u>exercício</u>

<u>Exemplo</u> $(\mathbb{R}^2)$:

$$[x_1 \; x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2$$

$$= \sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}\, x_j x_k$$

$$h''(0) = v^T \underbrace{Hf(a)}_{\boxed{\text{matriz hesseana de } f}} v$$

$$Hf(a) = \left[\frac{\partial^2 f}{\partial x_j \partial x_k}(a) \right]_{n \times n} \quad \leftarrow k = 1, \dots, n$$

$\uparrow j = 1, \dots, n$

Lema de Schwarz $\Rightarrow$ $Hf(a)$ é $\boxed{\text{SIMÉTRICA}}$

Exemplos:

$\mathbb{R}^2$ $\qquad\qquad$ $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} ; \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$\Rightarrow$ $Hf(a)$ é DIAGONALIZÁVEL :

$\lambda_1, \dots \lambda_n$ valores próprios REAIS

$\{v_1, \dots, v_n\}$ vectores próprios

base ortonormada de $\mathbb{R}^n$

$$h''(0) = v^T Hf(a)\, v$$

Basta analisar os vectores/valores próprios.

Se v for vector próprio de $Hf(a)$

tem-se: $Hf(a)\, v = \lambda v \,, \quad v \neq 0$
$\lambda \in \mathbb{R}$

$$\Rightarrow \underbrace{h''(0)}_{\text{sinal}} = v^T (\lambda v)$$

$$= \lambda v^T v$$

$$= \lambda\, v \cdot v$$

$$= \underbrace{\lambda}_{\text{sinal}!!!} \|v\|^2$$

Sinal de $h''(0)$ é o sinal do valor próprio λ.